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On Hubbard–Stratonovich transformations over hyperbolic domains

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Abstract
We discuss and prove the validity of the Hubbard–Stratonovich (HS) identities over hyperbolic domains which are used frequently in studies on disordered systems and random matrices. We also introduce a counterpart of the HS identity arising in disordered systems with 'chiral' symmetry. Apart from this we outline a way of deriving the nonlinear σ-model from the gauge-invariant Wegner k-orbital model avoiding the use of the HS transformations.

1. Introduction

A considerable progress achieved over the past two decades in understanding statistical properties of a single electron motion in disordered and chaotic systems [1, 2] is mainly based on the nonlinear σ-model description. This concept was originally proposed in the context of disordered systems by Wegner [3], and further clarified and developed to a working tool in the papers by Schäfer and Wegner [4] and Pruisken and Schäfer [5]. The authors reduced the problem of evaluating the disorder-averaged correlation functions of the resolvents (Green function) of a Hamiltonian containing disorder to an effective deterministic field theory with a very peculiar underlying non-compact symmetry, which they called 'hyperbolic'. Original derivations [4] used a specific microscopic model, Wegner's gauge-invariant k-orbital model[1]. It was, however, soon understood that the reduction is valid under much more general conditions. Loosely speaking, the nonlinear σ-model adequately describes physics of the one-electron motion for scales longer than the so-called mean free path. In the gauge-invariant model the mean free path is effectively zero, which makes the derivation particularly transparent.

In all the mentioned work the reduction of the microscopic disordered models to the nonlinear σ-model made use of the so-called replica limit. The last essential ingredient—the idea of supersymmetry, i.e. the use of both commuting and anticommuting variables to avoid the

[1] Readers not familiar with the k-orbital model and its relation to nonlinear σ-model may consult a recent paper [6], or appendix A of the present paper for a short introduction.

problematic replica method—was introduced in the theory by Efetov [1]. The resulting theory was successfully tested in an important non-perturbative limit of 'zero' spatial dimension, where its predictions were shown to be identical to those following from the theory of large random matrices. After that, within a few years the supersymmetric version of the nonlinear σ-model was accepted as a standard tool in condensed matter physics, and it also proved to be very useful in other fields, ranging from the theory of chaotic scattering [7, 8] to quantum chromodynamics [9].

Successful as it was in applications, the nonlinear σ-model is derived from underlying disordered Hamiltonian along a mathematically subtle procedure. One of the standard non-trivial ingredients of the derivation is the exploitation of the so-called Hubbard–Stratonovich identity[2]:

$$C_N \mathrm{e}^{-\frac{1}{2}\operatorname{Tr}\hat{A}^2} = \int \mathcal{D}\hat{R}\, \mathrm{e}^{-\frac{1}{2}\operatorname{Tr}\hat{R}^2 - \mathrm{i}\operatorname{Tr}\hat{R}\hat{A}} \tag{1}$$

where both $\hat{R}, \hat{A}$ are square $n \times n$ matrices with entries R_{ij} and A_{ij}, respectively, and C_N is some A-independent constant. When we take these matrices to be complex Hermitian (or real symmetric), use the corresponding 'flat differentials' $\mathcal{D}\hat{R} \propto \prod_{i=1}^{n} \mathrm{d}R_{ii} \prod_{i<j} \mathrm{d}\,\mathrm{Im}[R_{ij}]\,\mathrm{d}\,\mathrm{Re}[R_{ij}]$ as volume elements for the Hermitian case, and $\mathcal{D}\hat{R} \propto \prod_{i=1}^{n} \mathrm{d}R_{ii} \prod_{i<j} \mathrm{d}R_{ij}$ for the real symmetric case, and integrate in the right-hand side over all degrees of freedom independently from minus to plus infinity, the integral amounts to a product of n^2 (resp. $n(n+1)/2$) one-fold Gaussian integrals, and the identity is completely trivial. The tricky point is that in the problem under consideration such a simple choice of the integration manifold of matrices $\hat{R}$ is prohibited by some extra requirements which impose a restriction on the structure of the matrices $\hat{A}$ and $\hat{R}$. In particular, for the whole theory to be well-defined one has to ensure that the second (linear in $\hat{A}$) term in the exponent in the right-hand side is *purely imaginary*, by the very way one makes use of the HS identities. With a non-Hermitian choice of the matrices $\hat{A}, \hat{R}$ this is already a rather non-trivial task. For an informal discussion of this point in a simple example see [10]; a rigorous and comprehensive treatment can be found in the recent review paper by Zirnbauer [11].

The difficulty was first encountered and successfully solved by Schäfer and Wegner. For the model which stems on the microscopic level from disordered Hamiltonians with broken time-reversal invariance (e.g., due to the presence of a magnetic field) they suggested the following choice of the integration domain:

$$\hat{R} = \lambda \hat{T}\hat{T}^{\dagger} + \mathrm{i}\hat{P} \tag{2}$$

where $\hat{T} \in U(n_1, n_2)$ is in the pseudounitary group of complex $n \times n$ matrices, $n = n_1 + n_2$, with integer $n_1 \geqslant 1, n_2 \geqslant 1$. The inverse for such matrices is given by $\hat{T}^{-1} = \hat{L}\hat{T}^{\dagger}\hat{L}$, with $\hat{L} = \mathrm{diag}(\mathbf{1}_{n_1}, -\mathbf{1}_{n_2})$ where $\mathbf{1}_n$ stands for the identity matrix of size n, and $T^{\dagger}$ stands for the Hermitian conjugate of $\hat{T}$. The matrices $\hat{P}$ are Hermitian block-diagonal, $\hat{P} = \mathrm{diag}(\hat{P}_{n_1}, \hat{P}_{n_2}) = \hat{P}^{\dagger}$, and the parameter $\lambda > 0$ is arbitrary. For the case of Hamiltonians respecting the time-reversal invariance the structure of the integration manifold is very much the same, with the pseudoorthogonal group $O(n_1, n_2)$ replacing the pseudounitary one.

It is the manifold of matrices $\hat{T}$ that encapsulates the non-compact ('hyperbolic') symmetry of the problem, and at the next stage gives rise to the interacting Goldstone modes described by the nonlinear σ-model. With such a non-trivial choice of the integration manifold the identity (1) loses its transparency, and its verification is a separate, non-trivial task. The idea of the proof [4] is to interpret the corresponding integral as going over a high-dimensional contour

[2] The original paper [4] does not use the name Hubbard–Stratonovich, but such a terminology became standard after Efetov's work [1].

deformation of a simple Hermitian matrix. An accurate implementation of this argument can be found in the mentioned review paper by Zirnbauer [11].

Although the Schäfer–Wegner parameterization (2) of the integration manifold is admissible it was never much in use; it was virtually abandoned in favour of an alternative one due to Pruisken and Schäfer [5]:

$$\hat{R} = \hat{T}^{-1}\hat{P}\hat{T}, \qquad \mathcal{D}R = \mathrm{d}\mu_{\mathrm{H}}(T)\,\mathrm{d}P_1\,\mathrm{d}P_2\,\Delta^2[\hat{P}] \tag{3}$$

where $\hat{P} = \mathrm{diag}(\hat{P}_{n_1}, \hat{P}_{n_2})$, with $\hat{P}_{n_1}$ and $\hat{P}_{n_2}$ being real diagonal[3], and $\hat{T}$ is again as in equation (2). The notation $\mathrm{d}\mu_{\mathrm{H}}(T)$ is used here for the invariant (Haar's) measure on the pseudounitary group, and for any diagonal $n \times n$ matrix $\hat{B} = \mathrm{diag}(b_1, b_2, \ldots, b_n)$ we use the notation $\Delta[\hat{B}] = \prod_{i<j}(b_i - b_j)$ for the associated Vandermonde determinant. The matrices $\hat{A}$ relevant in present context are always of the form $\hat{A} = \hat{A}_+\hat{L}$, where $\hat{A}_+ \geqslant 0$ is Hermitian positive semidefinite, and $\hat{L}$ is the 'signature' matrix entering the definition of the pseudounitary group $U(n_1, n_2)$; see above. The Pruisken–Schäfer type of parameterization for the integration manifold is used in the majority of applications (in particular, in the supersymmetric version of the theory, implicitly in [1] and explicitly in [7] and subsequent papers). Nevertheless, it seems that validity of the corresponding Hubbard–Stratonovich transformation equation (1) was never properly checked. In fact, for many years it was taken for granted [10] that a kind of 'shift of the contour' argument should be valid for such a parameterization as well. The belief was however challenged by Zirnbauer (see criticism in [14], and recently in [11]) who revealed via a thorough analysis serious difficulties in implementing this type of argument for Pruisken–Schäfer parameterization. This observation makes the situation uncertain and calls for further investigations.

It is worth mentioning that recently an alternative method of treating the hyperbolic symmetry was introduced which avoids the use of any variant of the Hubbard–Stratonovich integral [12]; see a very informative discussion in [11]. This method can be successfully applied to the gauge-invariant k-orbital model, and we outline the corresponding derivation of the (bosonic) nonlinear σ-model in appendix A of the present paper (see also [6]). Nevertheless, the Hubbard–Stratonovich transformations remains an important (frequently, the only available) tool for the majority of microscopic models, in particular for the popular model of band matrices, see [13] and discussions in [11], as well as for the case of random potential ('diagonal') disorder.

The main goal of this paper is to provide a way of independent verification of the validity of the Hubbard–Stratonovich identity for the Pruisken–Schäfer type of parameterizations. We succeed in solving the problem in full generality for the pseudounitary case. As to the pseudoorthogonal case, the situation turns out to be more complicated and is in fact quite interesting. By considering explicitly the simplest non-trivial case $n_1 = n_2 = 1$ we will show that to save the identity one has to discard the properties for the factors entering the elementary integration volume to be positive and for the constant C_N to be real. Finally, we will discuss and prove the counterpart of the Hubbard–Stratonovich identity which arises naturally in the studies on disordered systems with special ('chiral') symmetry, and is also useful when investigating non-Hermitian random Hamiltonians. Some technical details are provided in appendices B and C.

[3] The convergence of the integral (5) in the text below requires, in fact, infinitesimal imaginary parts to be included in variables $\hat{P}$ as $\hat{P} = \mathrm{diag}(\hat{P}_{n_1} - \mathrm{i}0^{+}\mathbf{1}_n, \hat{P}_{n_2} + \mathrm{i}0^{+}\mathbf{1}_n)$. This shift is henceforth assumed implicitly. A similar remark is applicable also to the formula (25) in the 'chiral' case.

2. Hubbard–Stratonovich identities over Pruisken–Schäfer domains

2.1. Pseudounitary case

We start with proving (1) for the general case of the pseudounitary Pruisken–Schäfer domain $\hat{R} = \hat{T}^{-1}\hat{P}\hat{T}$ parameterized by matrices $\hat{T} \in U(n_1, n_2)$ and $\hat{P} = \mathrm{diag}(\hat{P}_1, \hat{P}_2)$ as in (3). It is easy to show (see e.g. [12], appendix A) that for strictly positive definite matrices $\hat{A}_+ > 0$ the corresponding matrices $\hat{A} = \hat{A}_+\hat{L}$ can always be parameterized as $\hat{A} = \hat{T}_A\hat{\Lambda}\hat{T}_A^{-1}$, where $\hat{\Lambda} = \mathrm{diag}(\hat{\Lambda}_{n_1}, \hat{\Lambda}_{n_2})$ is diagonal such that $\hat{\Lambda}_{n_1} > 0$, $\hat{\Lambda}_{n_2} < 0$, and $\hat{T}_A$ is a pseudounitary matrix from $U(n_1, n_2)$. Starting with this form of the matrix $\hat{A}$, we see that the right-hand side of (1) can be written as

$$I_{\mathrm{HS}}^{(\mathrm{pu})}(\hat{A}) = \int \mathrm{d}\hat{P}\, \Delta^2[\hat{P}]\, \mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\,\hat{P}^2} \int \mathrm{d}\mu_{\mathrm{H}}(\hat{T}) \exp\{-\mathrm{i}\,\mathrm{Tr}\,\hat{T}^{-1}\hat{P}\hat{T}\hat{T}_A\hat{\Lambda}\hat{T}_A^{-1}\}. \quad (4)$$

Now it is obvious that the integral is independent of the matrix $\hat{T}_A$. Indeed, that matrix can be absorbed into $\hat{T}$ by the change of variables $\hat{T}\hat{T}_A \to \hat{T}$, exploiting the cyclic invariance under the trace and the invariance property of the Haar's measure: $\mathrm{d}\mu_{\mathrm{H}}(T) \equiv \mathrm{d}\mu_{\mathrm{H}}(\hat{T}\hat{T}_A)$. A second observation is that due to the (block-)diagonal structure of the matrices $\hat{P}$, $\hat{\lambda}$ the combination in the exponent $\mathrm{Tr}[\hat{T}^{-1}\hat{P}\hat{T}\hat{\Lambda}]$ stays invariant when $\hat{T}$ is multiplied with an arbitrary unitary block-diagonal matrix of the form $\mathrm{diag}(\hat{V}_1, \hat{V}_2)$, with $\hat{V}_1 \in U(n_1)$ and $\hat{V}_2 \in U(n_2)$. Thus, the combination in question does not change if we replace $\hat{T}$ with matrices $\hat{T}_0 \in \frac{U(n_1,n_2)}{U(n_1)\otimes U(n_2)}$ taken from the coset space obtained by factorizing the original pseudounitary group $U(n_1, n_2)$ by its maximal compact subgroup $U(n_1) \otimes U(n_2)$. As the result, the integral—up to a multiplicative constant—can be replaced by one going over the coset space rather than the whole pseudounitary group. This is a very pleasing fact, since such an integral has been recently shown [15] to be exactly calculable with the help of the so-called Duistermaat–Heckman localization theorem, generalizing a similar formula known for the unitary group [19]. The result of such integration is given, again up to a multiplicative constant, by[4]

$$\int \mathrm{d}\mu(\hat{T}_0)\, \mathrm{e}^{-\mathrm{i}\,\mathrm{Tr}[\hat{T}_0^{-1}\hat{P}\hat{T}_0\hat{\Lambda}]} \propto \frac{\det\left[\mathrm{e}^{-\mathrm{i}p_{1i}\lambda_{1j}}\right]_{i,j=1}^{n_1} \det\left[\mathrm{e}^{-\mathrm{i}p_{2i}\lambda_{2j}}\right]_{i,j=n_1+1}^{n}}{\Delta[\hat{P}]\Delta[\hat{\Lambda}]}. \quad (5)$$

Substituting (5) back to (4), we bring the latter to the form

$$I_{\mathrm{HS}}^{(\mathrm{pu})}(\hat{A}) \propto \frac{1}{\Delta[\hat{\Lambda}]} \int \mathrm{d}\hat{P}_1\, \mathrm{d}\hat{P}_2\, \Delta[\hat{P}]\, \mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\,\hat{P}^2} \det\left[\mathrm{e}^{-\mathrm{i}p_{1i}\lambda_{1j}}\right]_{i,j=1}^{n_1} \det\left[\mathrm{e}^{-\mathrm{i}p_{2i}\lambda_{2j}}\right]_{i,j=n_1+1}^{n}. \quad (6)$$

Now we observe that $\Delta[\hat{P}] = \Delta[\hat{P}_1]\Delta[\hat{P}_2]\prod_{i=1}^{n_1}\prod_{j=n_1+1}^{n}(p_{1i} - p_{2j})$ and use the invariance of the integrand with respect to any permutation of the indices of integration variables in the set $p_{1i}, i = 1, \ldots, n_1$, as well as in the set $p_{2j}, j = n_1 + 1, \ldots, n$. Such an invariance allows one to replace each determinantal factor in the integrand with only one ('diagonal') contribution, multiplying the whole integral with the factor $n_1!n_2!$. Disregarding the multiplicative factors, we write the resulting integral as

$$I_{\mathrm{HS}}^{(\mathrm{pu})}(\hat{A}) \propto \frac{1}{\Delta[\hat{\Lambda}]} \int \mathrm{d}\hat{P}_1\, \mathrm{d}\hat{P}_2\, \Delta[\hat{P}]\, \mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\,\hat{P}^2} \mathrm{e}^{-\mathrm{i}\sum_{i=1}^{n_1} p_{1i}\lambda_{1i}} \mathrm{e}^{-\mathrm{i}\sum_{i=n_1+1}^{n} p_{2i}\lambda_{2i}} \quad (7)$$

and further use the following well-known identity[5]

$$\int \mathrm{d}\hat{P}\, \Delta[\hat{P}]\, \mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\,\hat{P}^2 \pm \mathrm{i}\,\mathrm{Tr}[\hat{P}\hat{\Lambda}]} \propto \Delta[\hat{\Lambda}]\, \mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\,\hat{\Lambda}^2}$$

[4] See the previous footnote on convergence of this integral.

[5] This formula can be for example derived starting from (1) for Hermitian matrices $\hat{R}$, $\hat{A}$, diagonalizing $\hat{R}$ by unitary transformation, and performing the unitary group integration by the Itzykson–Zuber–Harish-Chandra formula [19]. See appendix B for a proof of a similar expression in the 'chiral' case.

valid for any real diagonal matrices $\hat{P}$, $\hat{\Lambda}$. In view of $\mathrm{Tr}\,\hat{\Lambda}^2 \equiv \mathrm{Tr}\,\hat{A}^2$ the latter result shows that the original integral (4) is equal, up to a multiplicative A-independent constant, to $\mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\,\hat{A}^2}$. We thus proved the required Hubbard–Stratonovich identity for matrices $\hat{A} = \hat{A}_+\hat{L}$, with $\hat{A}_+ > 0$.

What remains to be shown is how to incorporate the case of positive semidefinite $\hat{A}_+ \geqslant 0$ in the above scheme, which is crucial for applications. The problem is that if the matrix $\hat{A}_+$ has zero eigenvalues, the corresponding matrix $\hat{A} = \hat{A}_+\hat{L}$ may not be T-diagonalizable[6]. To this end introduce a parameter $\varepsilon > 0$ and consider the integral

$$\int \mathcal{D}R\,\mathrm{e}^{-\frac{1}{2}\mathrm{Tr}(\hat{R}+\mathrm{i}\varepsilon\hat{L})^2 - \mathrm{i}\,\mathrm{Tr}\,\hat{A}\hat{R}} = \mathrm{e}^{\frac{1}{2}\varepsilon^2\,\mathrm{Tr}\,\mathbf{1}_n} \int \mathcal{D}R\,\mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\,\hat{R}^2 - \mathrm{i}\,\mathrm{Tr}(\hat{A}+\varepsilon\hat{L})\hat{R}}. \tag{8}$$

From $\hat{A} = \hat{A}_+\hat{L}$, where $\hat{A}_+ \geqslant 0$, it immediately follows that $\hat{A}_\epsilon \equiv \hat{A}+\varepsilon\hat{L} = (\hat{A}_+ + \varepsilon\mathbf{1}_n)\hat{L} \equiv \hat{A}_{\epsilon,+}\hat{L}$, where $\hat{A}_{\epsilon,+}$ is already positive definite: $\hat{A}_{\epsilon,+} = \hat{A}_+ + \varepsilon\mathbf{1}_n > 0$. But such matrices A_ϵ are always T-diagonalizable, and the above-given proof of the Hubbard–Stratonovich identity retains its validity. Therefore, the integral is calculated as

$$\mathrm{e}^{\frac{1}{2}\varepsilon^2\,\mathrm{Tr}\,\mathbf{1}_n} \int \mathcal{D}R\,\mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\,\hat{R}^2 - \mathrm{i}\,\mathrm{Tr}(\hat{A}_\epsilon\hat{R})} = \mathrm{e}^{\frac{1}{2}\varepsilon^2\,\mathrm{Tr}\,\mathbf{1}_n}\mathrm{e}^{-\frac{1}{2}\mathrm{Tr}(\hat{A}_\varepsilon)^2} = \mathrm{e}^{-\varepsilon\,\mathrm{Tr}\,\hat{L}\hat{A}}\mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\,\hat{A}^2}$$

resulting in the ϵ-modified version of the Hubbard–Stratonovich identity:

$$\int \mathcal{D}R\,\mathrm{e}^{-\frac{1}{2}\mathrm{Tr}(\hat{R}+\mathrm{i}\varepsilon\hat{L})^2 - \mathrm{i}\,\mathrm{Tr}\,\hat{A}\hat{R}} = \mathrm{e}^{-\varepsilon\,\mathrm{Tr}\,\hat{L}\hat{A}}\mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\,\hat{A}^2}. \tag{9}$$

For $\varepsilon > 0$ this identity holds uniformly in $\hat{A}$, including the case of non-diagonalizable matrices $\hat{A}$.

2.2. *Pseudoorthogonal case:* $n_1 = n_2 = 1$

The difficulty of proving (1) in the important case of general pseudoorthogonal group $O(n_1, n_2)$ is due to lack of integration formulae similar to equation (5) for cosets $\frac{\mathrm{O}(n_1,n_2)}{\mathrm{O}(n_1)\otimes O(n_2)}$. Under these circumstances we will restrict ourselves by the first non-trivial case $n_1 = n_2 = 1$, which proves to be already very informative. The matrix $\hat{P}$ in this case is 2×2 diagonal, $\hat{P} = \mathrm{diag}(p_1, p_2)$, and the matrices $\hat{T}_0$ can be explicitly parameterized in terms of the variable $\theta \in (-\infty, \infty)$ as $\hat{T}_0 = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$. Then the 2×2 matrices $\hat{R} = \hat{T}_0^{-1}\hat{P}\hat{T}_0$ we are integrating over in equation (1) are explicitly given by

$$\hat{R} = \begin{pmatrix} \frac{p_1+p_2}{2} + \frac{p_1-p_2}{2}\cosh 2\theta & \frac{p_1-p_2}{2}\sinh 2\theta \\ -\frac{p_1-p_2}{2}\sinh 2\theta & \frac{p_1+p_2}{2} - \frac{p_1-p_2}{2}\cosh 2\theta \end{pmatrix}. \tag{10}$$

As we already mentioned, the matrices $\hat{A}$ must be of the form $\hat{A} = \hat{A}_+\hat{L}$, where $\hat{L} = \mathrm{diag}(1, -1)$. We restrict ourselves in this section only with real symmetric matrices $\hat{A}_+ > 0$ for simplicity, i.e. $\hat{A}_+ = \begin{pmatrix} a_1 & a \\ a & a_2 \end{pmatrix} > 0$. Hence $\hat{A} = \begin{pmatrix} a_1 & -a \\ a & -a_2 \end{pmatrix}$, with the constraints

$$a_1 > 0, \qquad a_2 > 0, \qquad |a| < \sqrt{a_1 a_2}. \tag{11}$$

Naively, one may expect the choice of the volume element on such a manifold in the form $\mathrm{d}\hat{R} = |p_1 - p_2|\,\mathrm{d}p_1\,\mathrm{d}p_2\,\mathrm{d}\theta$ to be 'natural'. We will however see below that taking such a choice we end up in trouble: the corresponding Hubbard–Stratonovich formula (1) does not hold its validity any longer. Instead, the correct choice of the 'volume element' in our case turns out to be

$$\mathrm{d}\tilde{R} = (p_1 - p_2)\,\mathrm{d}p_1\,\mathrm{d}p_2\,\mathrm{d}\theta. \tag{12}$$

[6] The simplest relevant example is $\hat{A} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$, which squares to zero: $\hat{A}^2 = 0$.

This expression is not sign-definite any longer, and changes sign at the line $p_1 = p_2$; we will comment on this point shortly later on.

After having specified all ingredients of the right-hand side in (1) we can write down the corresponding integral explicitly as

$$I_{\mathrm{HS}}^{(\mathrm{po})} = \int_{-\infty}^{\infty} \mathrm{d}p_1 \int_{-\infty}^{\infty} \mathrm{d}p_2\,(p_1 - p_2)\,\mathrm{e}^{-\frac{1}{2}(p_1^2+p_2^2)-\mathrm{i}\frac{1}{2}(p_1+p_2)(a_1-a_2)} \int_{-\infty}^{\infty} \mathrm{d}\theta\,\mathrm{e}^{-\mathrm{i}\alpha\cosh 2\theta-\mathrm{i}\beta\sinh 2\theta} \quad (13)$$

where we introduced the shorthand notations $\alpha = \frac{1}{2}(a_1 + a_2)(p_1 - p_2)$, $\beta = a(p_1 - p_2)$. We note that in view of (11) it holds that $\beta/\alpha = \frac{a}{\frac{a_1+a_2}{2}} \leqslant \frac{a}{\sqrt{a_1a_2}} < 1$. Therefore, we can parameterize $\beta = u\sinh\psi$, $\alpha = u\cosh\psi$, where u, ψ are real parameters. Then the combination entering the second exponent in (13) can be rewritten as $\alpha\cosh 2\theta + \beta\sinh 2\theta \equiv u\cosh(2\theta + \psi)$. Finally, introducing $\mu = 2\theta + \psi$ as an integration variable, we see that the integral over θ can be explicitly calculated as [16](a):

$$\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}\mu \mathrm{e}^{-\mathrm{i}u\cosh\mu} = -\frac{\pi}{2}[Y_0(|u|) + \mathrm{i}\,\mathrm{sgn}(u) J_0(|u|)] \equiv K_0(\mathrm{i}u) \quad (14)$$

where $\mathrm{sgn}(u) = \pm 1$ depending on the sign of the variable u, and $J_0(z), Y_0(z), K_0(z)$ are Bessel, Neumann and Macdonald functions of zero order, respectively [16]. In our case,

$$u = (p_1 - p_2)s_a, \qquad s_a \equiv \sqrt{\left(\frac{a_1 + a_2}{2}\right)^2 - a^2}. \quad (15)$$

Substituting the result of integration back to (13) and changing to integration variables $p_\pm = (p_1 \pm p_2)$, one can easily perform the Gaussian integral over p_+ and obtain

$$I_{\mathrm{HS}}^{(\mathrm{po})} = -\mathrm{i}\pi^{3/2}\mathrm{e}^{-\frac{1}{4}(a_1-a_2)^2} \int_{-\infty}^{\infty} \mathrm{d}p_-\, p_- \mathrm{e}^{-\frac{1}{4}p_-^2}[Y_0(|p_-|s_a) + \mathrm{i}\,\mathrm{sgn}(p_-) J_0(|p_-|s_a)]. \quad (16)$$

This is exactly the point where we can most clearly see the necessity of omitting the absolute value sign in the measure (12). Indeed, had we maintained the modulus $|p_-|$ in the above integral, the second term in the integrand, being odd in p_-, would vanish and the remaining integral would be that containing the Neumann function. Although it is well-defined, it could not produce the structure necessary for the validity of the identity equation (1). In contrast, when the factor p_- in the measure does not contain the absolute value, it is the first term which vanishes, and the second term can be straightforwardly integrated by using the identity [16](b):

$$\int_0^{\infty} \mathrm{d}p\, p\mathrm{e}^{-bp^2} J_0(pc) = \frac{1}{2b}\mathrm{e}^{-\frac{c^2}{4b}}$$

yielding, with $b \equiv 1/4$, $c \equiv s_a$

$$I_{\mathrm{HS}}^{(\mathrm{po})} = -4\mathrm{i}\pi^{3/2}\mathrm{e}^{-\frac{1}{4}(a_1-a_2)^2-\frac{1}{4}(a_1+a_2)^2+a^2} = -4\mathrm{i}\pi^{3/2}\mathrm{e}^{-\frac{1}{2}(a_1^2+a_2^2-2a^2)} \equiv -4\mathrm{i}\pi^{3/2}\mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\,\hat{A}^2} \quad (17)$$

exactly as required by the Hubbard–Stratonovich identity equation (1).

The considered example (and also one in the next section) makes it clear that the only consistent way of ensuring the validity of the HS transformation over hyperbolic domains is to require the absence of the absolute value of non-positively defined factors in the elementary volume. From a geometric point of view the integration domain in the hyperbolic case consists of several (in the simplest case two) disconnected pieces. One may notice that the factor in the elementary volume (12) changes sign precisely when passing from one such piece to a different one, being sign-constant within each piece. Persistence of this structure for $n > 1$, as well as finding a comprehensive proof valid for any n in the pseudoorthogonal case, remains an interesting open question deserving further attention.

2.3. Chiral variant of the Hubbard–Stratonovich identity

In the last decade, new symmetry classes of random Hamiltonians attracted a lot of interest due to numerous applications in various branches of physics; see [17] for a discussion and basic references. In particular, the class of Hamiltonians with chiral symmetry is pertinent for analysing properties of Dirac fermions in random gauge field background, and found applications in quantum chromodynamics [9], as well as in condensed matter theory; see e.g. [18] for more references and discussion. When reducing the analysis of such systems to the relevant nonlinear σ-model, one encounters the following variant of the Hubbard–Stratonovich identity:

$$I_{\mathrm{HS}}^{(\mathrm{ch})}(\hat{A},\hat{B}) = C_N\,\mathrm{e}^{-\mathrm{Tr}[\hat{A}\hat{B}]} = \int \mathcal{D}\hat{R}_1\,\mathcal{D}\hat{R}_2\,\mathrm{e}^{-\mathrm{Tr}\,\hat{R}_1\hat{R}_2 - \mathrm{i}\,\mathrm{Tr}[\hat{R}_1\hat{A}+\hat{B}\hat{R}_2]}. \tag{18}$$

In the simplest case the two involved matrices are related as $\hat{A}^\dagger = \hat{B} \in GL(n,\mathcal{C})$, i.e. $\hat{A}$ is an arbitrary complex matrix, and one can make a natural choice of the integration domain $\hat{R}_1^\dagger = \hat{R}_2 \in GL(n,\mathcal{C})$, with elementwise 'flat measure' on it. The linear in $\hat{A}$ term in the exponent is then purely imaginary as required, and the identity

$$I_{\mathrm{HS}}^{(\mathrm{ch})}(\hat{A},\hat{A}^\dagger) = C_N\,\mathrm{e}^{-\mathrm{Tr}[\hat{A}\hat{A}^\dagger]} = \int \mathcal{D}\hat{R}\,\mathcal{D}\hat{R}^\dagger\,\mathrm{e}^{-\mathrm{Tr}\,\hat{R}^\dagger\hat{R} - \mathrm{i}\,\mathrm{Tr}[\hat{R}^\dagger\hat{A}+\hat{A}^\dagger\hat{R}]} \tag{19}$$

follows from the integral in the right-hand side being the standard Gaussian one.

In the applications, however, the case of two unrelated Hermitian positive semidefinite matrices $\hat{A}^\dagger = \hat{A} \geqslant 0$, $\hat{B}^\dagger = \hat{B} \geqslant 0$ emerges as well, and the extra convergency arguments necessitate making the choice for the integration domain to be compatible with that property (see below). Again, ensuring the pure imaginary nature of the exponent and subsequent verification of the Hubbard–Stratonovich identity (18) for such a domain is a non-trivial task which has not, to the best of author's knowledge, yet been accomplished in full generality.

Our strategy in this case will be informed both by our experience with the pseudounitary and pseudoorthogonal cases. Again restricting ourselves in this section with $\hat{A} > 0$, $\hat{B} > 0$, we observe that any pair of Hermitian, positive definite $n \times n$ matrices $\hat{A}$, $\hat{B}$ can be parameterized as (see [18] appendix A)

$$\hat{A} = \hat{T}_A\hat{a}\hat{T}_A^\dagger, \qquad \hat{B} = [\hat{T}_A^\dagger]^{-1}\hat{a}\hat{T}_A^{-1} \tag{20}$$

where

$$\hat{a} = \mathrm{diag}(a_1,\ldots,a_n) > 0, \qquad \hat{T}_A \in \frac{GL(n,\mathcal{C})}{U(1)\times\cdots\times U(1)}, \tag{21}$$

that is the matrix $\hat{T}_A$ is a general complex matrix with real positive diagonal entries. This suggest an idea to parameterize the integration manifold as

$$\hat{R}_1 = [\hat{T}^\dagger]^{-1}\hat{P}[\hat{T}]^{-1}, \qquad \hat{R}_2 = \hat{T}\hat{P}\hat{T}^\dagger \tag{22}$$

in terms of a real diagonal matrix $-\infty < \hat{P} = \mathrm{diag}(p_1,\ldots,p_n) < \infty$ and a general complex matrix $\hat{T} \in \mathsf{GL}(\mathsf{n},\mathcal{C})$. Guided by our previous experience, the volume element in the new coordinates is chosen by

$$\mathcal{D}\hat{R}_1\,\mathcal{D}\hat{R}_2 \propto \prod_{l=1}^{n} p_l\,\mathrm{d}p_l \prod_{l<m}(p_l^2 - p_m^2)^2\,\mathrm{d}\mu_{\mathrm{H}}(\hat{T},\hat{T}^\dagger) \tag{23}$$

where $\mathrm{d}\mu_{\mathrm{H}}(T,T^\dagger)$ is the invariant Haar's measure on the group $\mathsf{Gl}(\mathsf{n},\mathcal{C})$. Note that despite the domain of integration being $-\infty < p_i < \infty$, $i = 1,\ldots,n$ the volume element contains factors p_i rather than $|p_i|$ as one may naively expect.

Substituting such a parameterization into the integral (18) and using the cyclic invariance under the trace, we have $\mathrm{Tr}[\hat{R}_1\hat{R}_2] = \mathrm{Tr}\,\hat{P}^2$, and the T-dependent term in the exponent is given by

$$\mathrm{Tr}[\hat{R}_1\hat{A} + \hat{B}\hat{R}_2] = \mathrm{Tr}\,\hat{P}[\hat{T}^\dagger\hat{T}_A\hat{a}\hat{T}_A^\dagger\hat{T} + \hat{T}^{-1}(\hat{T}_A^\dagger)^{-1}\hat{a}\hat{T}_A^{-1}(\hat{T}^\dagger)^{-1}].$$

Now we change variables $\hat{T}_A^\dagger\hat{T} \to \tilde{T}$, and exploiting the invariance of the measure $\mathrm{d}\mu_{\mathrm{H}}(T, T^\dagger) = \mathrm{d}\mu_{\mathrm{H}}(\tilde{T}, \tilde{T}^\dagger)$ satisfy ourselves that we need to deal with the following group integral:

$$\int_{T\in \mathsf{Gl}(\mathsf{n},\mathcal{C})} \mathrm{d}\mu_{\mathrm{H}}(T, T^\dagger)\exp -\mathrm{i}\,\mathrm{Tr}\,\hat{P}[\hat{T}^\dagger\hat{a}\hat{T} + \hat{T}^{-1}\hat{a}(\hat{T}^\dagger)^{-1}]. \tag{24}$$

Again, due to the diagonal structure of the matrices $\hat{P}, \hat{a}$ the integrand is not changed if we replace matrices $\hat{T} \in \mathsf{Gl}(\mathsf{n}, \mathcal{C})$ with $\hat{T}_0 \in \frac{GL(n,\mathcal{C})}{U(1)\times\cdots\times U(1)}$. This fact provides us with the possibility of exploiting one more integration formula discovered in [18][7]:

$$\int_{T\in\frac{\mathsf{Gl}(\mathsf{n},\mathcal{C})}{[\mathsf{U}(1)\times\cdots\times\mathsf{U}(1)]}} \mathrm{d}\mu_0(T_0, T_0^\dagger)\,\mathrm{e}^{-\mathrm{i}\,\mathrm{Tr}\,\hat{P}[\hat{T}_0^\dagger\hat{a}\hat{T}_0 + \hat{T}_0^{-1}\hat{a}(\hat{T}_0^\dagger)^{-1}]} \propto \frac{\det\left[K_0(2\mathrm{i}p_i a_j)\right]_{i,j=1}^n}{\Delta[\hat{P}^2]\Delta[\hat{a^2}]} \tag{25}$$

where $\mathrm{d}\mu_0(\hat{T}_0, \hat{T}_0^\dagger) = \mathrm{d}\hat{T}_0\,\mathrm{d}\hat{T}_0^\dagger\,\det\left[\hat{T}_0\hat{T}_0^\dagger\right]^{-n+\frac{1}{2}}$ is exactly the volume element on the coset space manifold. Substituting this result back to the integral (18), cancelling one of the Vandermonde factors coming from the volume element (23), and again exploiting the invariance properties of the integrand with respect to permutation of integration variables in the set $p_1, \ldots, p_n$ (cf discussion after (6)), and disregarding A-independent multiplicative factors, we bring the integral of (18) to the form

$$I_{\mathrm{HS}}^{(\mathrm{ch})}(\hat{A}, \hat{B}) \propto \frac{1}{\Delta^2[\hat{a}^2]}\int_{-\infty}^{\infty}\prod_{l=1}^n \mathrm{d}p_l\,p_l\Delta[\hat{P}^2]\,\mathrm{e}^{-\sum_{l=1}^n p_l^2}\prod_{l=1}^n K_0(2\mathrm{i}p_l a_l) \tag{26}$$

$$\propto \frac{1}{\Delta^2[\hat{a}^2]}\sum_{S_\alpha}(-1)^{S_\alpha}\prod_{l=1}^n\left[\int_{-\infty}^{\infty}\mathrm{d}p\,p\mathrm{e}^{-p^2}p^{2s_l}K_0(2\mathrm{i}pa_l)\right] \tag{27}$$

where we used the expansion of the Vandermonde determinant as $\Delta[\hat{P}^2] = \sum_{S_\alpha}(-1)^{S_\alpha}\prod_{l=1}^n p^{2s_l}$ in terms of the sum over $n!$ permutations $S_\alpha = (s_1, s_2, \ldots, s_n)$ of the index set $(1, 2, \ldots, n)$, with $(-1)^{S_\alpha}$ standing for the parity of the permutation. Now we substitute the formula (14) for the Bessel function of the imaginary argument to the above expression, and see again that the term with the Neumann function is multiplied with the odd in p factor and vanishes upon integration, yielding

$$I_{\mathrm{HS}}^{(\mathrm{ch})}(\hat{A}, \hat{B}) \propto \frac{1}{\Delta^2[\hat{a}^2]}\sum_{S_\alpha}(-1)^{S_\alpha}\prod_{l=1}^n\left[(-\mathrm{i}\pi)\int_0^{\infty}\mathrm{d}p\,p\mathrm{e}^{-p^2}p^{2s_l}J_0(2pa_l)\right] \tag{28}$$

where we used $\mathrm{sign}(a_i) = 1$. Finally, we invert the operation of the Vandermonde determinant expansion and show in appendix B that the resulting integral can be evaluated as

$$\frac{1}{\Delta^2[\hat{a}^2]}\int_0^{\infty}\prod_{l=1}^n \mathrm{d}p_l\,p_l\Delta[\hat{P}^2]\,\mathrm{e}^{-\sum_{l=1}^n p_l^2}\prod_{l=1}^n J_0(2p_l a_l) \propto \mathrm{e}^{-\sum_{l=1}^n a_l^2} = \mathrm{e}^{-\mathrm{Tr}\,\hat{A}\hat{B}}, \tag{29}$$

exactly as required by the Hubbard–Stratonovich identity (18).

[7] Although the formula itself is correct, its derivation in appendix B of [18] was not accurate enough. For this reason we include the outline of the correct derivation in appendix C to this paper.

Acknowledgments

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Appendix A. From k-orbital model to nonlinear σ-model without Hubbard–Stratonovich identity

Consider the $N \times N$ random Hermitian matrix $\hat{H}$, with all its entries H_{lm} for $l \leqslant m$ being independent random Gaussian with the variance $\langle H_{lm}^* H_{lm} \rangle = J_{lm} > 0$. Let $N = rk$, with r and k being integers. Subdivide the index set $I = 1, 2, \ldots, N$ into r subsets $I_1, I_2, \ldots, I_r$ each having exactly k elements, $I_i = \{(i-1)k+1, (i-1)k+2, \ldots, (i-1)k+k\}$, and consider the variances J_{lm} such that

$$J_{lm} = \begin{cases} J/k, & \text{if } (l, m \in I_i) \\ V/k^2, & \text{if } (l \in I_i, m \in I_{i+1}) \quad \text{or} \quad (l \in I_{i-1}, m \in I_i) \\ 0 & \text{otherwise.} \end{cases} \tag{A.1}$$

This defines the Wegner gauge-invariant k-orbital model [4] of r 'sites' arranged in the one-dimensional lattice $d = 1$. Each 'site' is a block I_i representing a group of k orbitals, with J/k standing for intragroup coupling and V/k^2 representing the coupling of neighbouring groups. The matrix $\hat{H}$ can be visualized as having a (block-)banded structure made of $k \times k$ blocks, with non-zero entries concentrated in a band of the widths $\propto k$ around the main diagonal. The couplings are scaled in a way ensuring correct behaviour in the limit of infinitely many orbital $k \to \infty$, which is assumed at a later stage (see also the closely related model in [23]). Generalization to lattices of higher spatial dimensions is obvious.

Our goal here is to demonstrate that the method developed in [12] for random matrix problems without underlying lattice structure can be straightforwardly applied to arrive at a nonlinear σ-model from the $\hat{H}$ defined above. We start with considering the simplest non-trivial object, the negative integer moment of the modulus of the spectral determinant of $\hat{H}$:

$$\mathcal{Z}_{n,n}(E, \eta) = \langle |\det(E + \mathrm{i}\eta/k - \hat{H})|^{-2n} \rangle_{\mathrm{H}} \tag{A.2}$$

where E and $\eta > 0$ are real and brackets stand for the ensemble averaging. Our first goal is to show that it has the following integral representation:

$$\mathcal{Z}_{n,n}(E, \eta) = \text{const} \times \int \mathrm{d}\hat{q}_1 \cdots \int \mathrm{d}\hat{q}_r \, \mathrm{e}^{-k \sum_{i=1}^{r} \mathcal{L}(\hat{q}_i)} \\ \times \prod_{i=1}^{r} \frac{1}{[\det \hat{q}_i]^{2n}} \mathrm{e}^{-\frac{V}{2} \sum_{i=1}^{r-1} \mathrm{Tr}[\hat{q}_i \hat{L} \hat{q}_{i+1} \hat{L}] - \eta \sum_{i=1}^{r} \mathrm{Tr}\, \hat{q}_i} \tag{A.3}$$

where $\hat{L} = \mathrm{diag}(\mathbf{1}_n, -\mathbf{1}_n)$

$$\mathcal{L}(\hat{q}) = \frac{J}{2} \mathrm{Tr}\, \hat{q}^2 - \mathrm{i}E\, \mathrm{Tr}\, \hat{q} - \mathrm{Tr} \ln \hat{q} \tag{A.4}$$

and the integration goes over the positive definite Hermitian $2n \times 2n$ matrices $\hat{q}_i > 0$. This representation is exact (no approximations) with the only restriction $k \geqslant 2n$.

Here we outline the main steps employed to derive equation (A.3). For every $l = 1, \ldots, N$ introduce two n-component vectors $\Phi_{l,+}, \Phi_{l,-}$ with complex components $S^{(1)}_{l,\pm}, \ldots, S^{(n)}_{l,\pm}$ and also use the notation $\Psi_l = \binom{\Phi_{l,+}}{\Phi_{l,-}}$ and $\eta_k = \eta/k$. Representing the inverse determinants as Gaussian integrals over $\Phi_{l,\pm}$ we have

$$\mathcal{Z}_{n,n}(E,\eta) \propto \int \mathrm{d}\Psi_1 \cdots \int \mathrm{d}\Psi_N \,\exp\left\{\sum_{l=1}^{N} \Psi_l^\dagger(\mathrm{i}E\hat{L} - \eta_k \mathbf{1}_{2n})\Psi_l\right\}\left\langle \mathrm{e}^{-\mathrm{i}\sum_{lm} H_{lm}\Psi_l^\dagger \hat{L}\Psi_m}\right\rangle_{\mathrm{H}}. \tag{A.5}$$

Now for any $l = 1, \ldots, N$ we introduce $2n \times 2n$ Hermitian matrices $\hat{Q}_l = \Psi_l \otimes \Psi_l^\dagger$, so that the result of the (Gaussian) averaging for any variances J_{lm} can be written as

$$\left\langle \mathrm{e}^{-\mathrm{i}\sum_{lm} H_{lm}\Psi_l^\dagger \hat{L}\Psi_m}\right\rangle_{\mathrm{H}} = \mathrm{e}^{-\frac{1}{2}\sum_{lm} J_{lm}\,\mathrm{Tr}(\hat{Q}_l\hat{L}\hat{Q}_m\hat{L})}.$$

For the particular choice of the variances equation (A.1) we can further write

$$\sum_{lm} J_{lm}\,\mathrm{Tr}(\hat{Q}_l\hat{L}\hat{Q}_m\hat{L}) = \frac{J}{k}\sum_{i=1}^{r}\mathrm{Tr}\left(\sum_{s=1}^{k}\hat{Q}_s^{(i)}\hat{L}\right)^2 + \frac{V}{k^2}\sum_{i=1}^{r-1}\mathrm{Tr}\left(\sum_{s=1}^{k}\hat{Q}_s^{(i)}\hat{L}\right)\left(\sum_{s=1}^{k}\hat{Q}_s^{(i+1)}\hat{L}\right)$$

where we denoted $\hat{Q}_s^{(i)} \equiv \hat{Q}_{k(i-1)+s}$. Introduce now the set of matrices $\hat{q}_i = \sum_{s=1}^{k}\hat{Q}_s^{(i)}$ for $i = 1, \ldots, r$ and rewrite the integral as

$$\mathcal{Z}_{n,n}(E,\eta) \propto \int \mathrm{d}\hat{q}_1 \cdots \mathrm{d}\hat{q}_r\, \mathcal{I}(\hat{q}_1)\cdots\mathcal{I}(\hat{q}_r)\,\mathrm{e}^{-\frac{J}{2k}\mathrm{Tr}\sum_{i=1}^{r}(\hat{q}_i\hat{L})^2 - \frac{V}{2k^2}\sum_{i=1}^{r-1}\mathrm{Tr}(\hat{q}_i\hat{L})(\hat{q}_{i+1}\hat{L})} \tag{A.6}$$

where

$$\mathcal{I}(\hat{q}) = \int \mathrm{d}\Psi_1\cdots\mathrm{d}\Psi_k\,\delta\left(\hat{q} - \sum_{s=1}^{k}\Psi_s\otimes\Psi_s^\dagger\right)\exp\left\{\sum_{s=1}^{k}\Psi_s^\dagger(\mathrm{i}E\hat{L} - \eta_k\mathbf{1}_{2n})\Psi_s\right\}. \tag{A.7}$$

Representing the δ-function factor as the Fourier transformation over the Hermitian $2n \times 2n$ matrix $\hat{K}$,

$$\delta\left(\hat{q} - \sum_{s=1}^{k}\Psi_s\otimes\Psi_s^\dagger\right) \propto \int \mathrm{d}\hat{K}\,\mathrm{e}^{\mathrm{i}\,\mathrm{Tr}\,\hat{K}\hat{q}}\,\mathrm{e}^{-\mathrm{i}\sum_{s=1}^{k}\mathrm{Tr}[\hat{K}(\Psi_s\otimes\Psi_s^\dagger)]}$$

and taking into account $\mathrm{Tr}[\hat{K}(\Psi_s\otimes\Psi_s^\dagger)] = \Psi_s^\dagger\hat{K}\Psi_s$, we find that the Ψ-integrals are Gaussian (and convergent in view of $\eta_k > 0$) and so when performed explicitly yield the factor

$$\det[\hat{K} - E\hat{L} - \mathrm{i}\eta_k\mathbf{1}_{2n}]^{-k}.$$

We immediately see that the resulting integral over $\hat{K}$ is precisely one of Ingham–Siegel type calculated in [12][8]. For $k \geqslant 2n$ this gives

$$\mathcal{I}(\hat{q}) = \theta(\hat{q})(\det\hat{q})^{k-2n}\mathrm{e}^{\mathrm{i}\,\mathrm{Tr}\,\hat{q}(E\hat{L}+\mathrm{i}\eta_k\mathbf{1}_{2n})} \tag{A.8}$$

where the factor $\theta(\hat{q})$ is non-zero for positive definite matrices and zero otherwise. Substitute this expression back to equation (A.6) and finally change $\hat{q} \to k\hat{q}$ This immediately produces the formula (A.3).

Now change the integration variables in equation (A.3) from $\hat{q}_i$ for $i = 1, \ldots, r$ to the matrices $\hat{\sigma}_i = \hat{q}_i\hat{L}$ parameterized as $\hat{\sigma}_i = \hat{T}_i^{-1}\hat{P}_i\hat{T}_i$, with $\hat{T}_i \in U(n,n)/U(1)\otimes\cdots\otimes U(1)$ and $\hat{P}_i = \mathrm{diag}(p^{(1)}_{1,i}, \ldots, p^{(1)}_{n,i}, p^{(2)}_{1,i}, \ldots, p^{(2)}_{n,i})$, such that $p^{(1)}_{l,i} > 0$, $p^{(2)}_{l,i} < 0$. Correspondingly,

[8] A mathematically-minded reader who may dislike a little bit frivolous manipulations with δ-functions may wish to perform the derivation by exploiting the 'integration theorem' proved in [15]. See also an alternative way of derivation in [6].

for each $i = 1, \ldots, r$ the integration measure is given by $\mathrm{d}\hat{\sigma}_i \propto \Delta^2(\hat{P}_i)\,\mathrm{d}\mu(T_i)\,\mathrm{d}\hat{P}_{1,i}\,\mathrm{d}\hat{P}_{2,i}$ and the above integral assumes the form

$$\mathcal{Z}_{n,n}(E,\eta) = \text{const} \times \int \mathrm{d}\hat{\sigma}_1 \cdots \int \mathrm{d}\hat{\sigma}_r\, \mathrm{e}^{-k\sum_{i=1}^{r}\mathcal{L}(\hat{P}_i)} \times \prod_{i=1}^{r} \frac{1}{[\det \hat{P}_i]^{2n}} \mathrm{e}^{-\frac{V}{2}\sum_{i=1}^{r-1}\operatorname{Tr}\hat{\sigma}_i\hat{\sigma}_{i+1} - \eta\sum_{i=1}^{r}\operatorname{Tr}\hat{\sigma}_i\hat{L}} \tag{A.9}$$

where we denoted

$$\mathcal{L}(\hat{P}) = \frac{J}{2}\operatorname{Tr}\hat{P}^2 - \mathrm{i}E\operatorname{Tr}\hat{P} - \operatorname{Tr}\ln\hat{P}. \tag{A.10}$$

Now it is evident that in the limit $k \to \infty$ all the matrices $\hat{P}_i$ will be fixed by the (unique) saddle-point value $\hat{P}_0$ minimizing the 'action' $\mathcal{L}(\hat{P})$ and given by

$$\hat{P}_0 = \frac{1}{2J}(\mathrm{i}E\hat{\mathbf{1}}_{2n} + \hat{L}\sqrt{4J - E^2}). \tag{A.11}$$

At the next step one should accurately perform the integration of Gaussian fluctuations around the saddle-point (the presence of the Vandermonde determinants makes the pre-exponential factors vanishing at the saddle-point for any $n \geqslant 2$). The resulting expression will clearly have the following structure:

$$\mathcal{Z}_{n,n}(E,\eta) \propto \int \mathrm{d}\mathcal{Q}_1 \cdots \int \mathrm{d}\mathcal{Q}_r\{\cdots\}\, \mathrm{e}^{-\frac{V}{2J}\sum_{i=1}^{r-1}\operatorname{Tr}\mathcal{Q}_i\mathcal{Q}_{i+1} - \frac{\eta}{\sqrt{J}}\sum_{i=1}^{r}\operatorname{Tr}\mathcal{Q}_i\hat{L}} \tag{A.12}$$

where the integration manifold is parameterized by the matrices $\mathcal{Q}_i = \hat{T}_i^{-1}\hat{L}\hat{T}_i$. Here we used dots to denote pre-exponential factors which may possibly arise and also set the energy parameter $E = 0$ for simplicity. This is exactly the lattice version of the nonlinear σ-model introduced by Wegner and Schäfer [4]. The index $i = 1, \ldots, r$ numbers the lattice sites and for $r \to \infty$ in the lattices of dimensions $d > 2$ the effective coupling constant $\frac{V}{2J}$ controls the transition from localized to extended states in the underlying Hamiltonian H.

The saddle-point procedure at $k \to \infty$ and subsequent manipulations are still to be done rigorously by strict mathematical standards; see recent progress and discussions of related issues in [6, 24]. Another interesting problem is how to include anticommuting degrees of freedom in the above derivation. Technically this can be done following various methods, and will be discussed elsewhere [25]; see also [11].

Appendix B. Proof of the formula (29)

The starting point of the proof is the identity (19). Introduce the singular value decompositions of the general complex matrices $\hat{R}$, $\hat{A}$ as

$$\begin{aligned} \hat{R} &= \hat{U}\hat{P}\hat{V}^\dagger, & \hat{P} &= \operatorname{diag}(p_1, \ldots, p_n) > 0, \\ \hat{A} &= \hat{U}_A\hat{a}\hat{V}_A^\dagger, & \hat{a} &= \operatorname{diag}(a_1, \ldots, a_n) > 0 \end{aligned} \tag{B.1}$$

where $\hat{U}$, $\hat{U}_A$, $\hat{V}$, $\hat{V}_A$ are $n \times n$ unitary, with the associated Haar's measures $\mathrm{d}\mu(\hat{U})$ and $\mathrm{d}\mu(\hat{V})$, respectively. The volume element $\mathcal{D}\hat{R}\mathcal{D}\hat{R}^\dagger$ in new coordinates associated with the singular value decomposition is given by $\mathcal{D}\hat{R}\mathcal{D}\hat{R}^\dagger \propto \prod_{l=1}^{n} p_l\,\mathrm{d}p_l \prod_{l<m}(p_l^2 - p_m^2)^2\,\mathrm{d}\mu(\hat{U})\,\mathrm{d}\mu(\hat{V})$.

Introducing $\tilde{U} = \hat{U}^\dagger\hat{U}_A$ and $\tilde{V}^\dagger = \hat{V}_A^\dagger\hat{V}$, and using the invariance of the Haar's measures, $\mathrm{d}\mu(\hat{U}) = \mathrm{d}\mu(\tilde{U})$, $\mathrm{d}\mu(\hat{V}) = \mathrm{d}\mu(\tilde{V})$, we rewrite the integral in the right-hand side of (19) as

$$\int_0^\infty \prod_{l=1}^{n} \mathrm{d}p_l\, p_l \Delta^2[\hat{P}^2]\, \mathrm{e}^{-\sum_{l=1}^{n} p_l^2} \int \mathrm{d}\mu(\tilde{U}) \int \mathrm{d}\mu(\tilde{V})\, \mathrm{e}^{-\mathrm{i}\operatorname{Tr}[\hat{P}(\tilde{U}\hat{a}\tilde{V}^\dagger + \tilde{V}\hat{a}\tilde{U}^\dagger)]}. \tag{B.2}$$

The integral over two unitary matrices in this expression is well-known from the papers by Guhr and Wettig [20] and Jackson *et al* [21]:

$$\int \mathrm{d}\mu(\hat{U})\,\mathrm{d}\mu(\hat{V})\,\mathrm{e}^{-\mathrm{i}\,\mathrm{Tr}(\hat{P}[\hat{U}\hat{a}_d\hat{V}^\dagger+\hat{V}\hat{a}_d\hat{U}^\dagger])} \propto \frac{\det[J_0(2p_ia_j)]|_{1\leqslant i,j\leqslant n}}{\Delta(p_1^2,\ldots,p_n^2)\Delta(a_1^2,\ldots,a_n^2)}. \tag{B.3}$$

Now substitute (B.3) into (B.2), and after cancelling one Vandermonde factor observe that due to the invariance of the integrand all $n!$ terms of the determinant made of Bessel functions yield identical contributions. Disregarding the multiplicative constants one can therefore replace that determinant with $\prod_i^n J_0(2p_ia_i)$, so that the right-hand side of (19) takes the form

$$\frac{1}{\Delta[\hat{a}^2]}\int_0^\infty \prod_{l=1}^n \mathrm{d}p_l\, p_l\Delta[\hat{P}^2]\,\mathrm{e}^{-\sum_{l=1}^n p_l^2}\prod_l^n J_0(2p_la_l). \tag{B.4}$$

On the other hand, the left-hand side is $\exp -\mathrm{Tr}\,\hat{A}^\dagger\hat{A} \equiv \mathrm{e}^{-\sum_{l=1}^n a_l^2}$, which proves the formula (29).

Appendix C. Matrix Macdonald functions associated with integrals over complex matrices

The integral (B.3) can be looked at as a certain matrix Bessel function that corresponds to Itzykson–Zuber-like integrals over unitary matrices; see details in Guhr and Wettig, Guhr and Kohler [20, 22]. Below we consider the matrix Macdonald functions that are associated with integrals over arbitrary complex matrices:

$$\int \mathrm{d}\mu(\hat{T},\hat{T}^\dagger)\,\mathrm{e}^{-\frac{1}{2}\mathrm{Tr}(\hat{X}_d[\hat{T}\hat{Y}_d\hat{T}^\dagger+(\hat{T}^\dagger)^{-1}\hat{Y}_d\hat{T}^{-1}])} = \mathrm{const}\frac{\det[K_0(x_iy_j)]|_{1\leqslant i,j\leqslant n}}{\Delta(x_1^2,\ldots,x_n^2)\Delta(y_1^2,\ldots,y_n^2)} \tag{C.1}$$

where $\hat{X}_d$ and $\hat{Y}_d$ are two positive definite diagonal matrices:

$$\hat{X}_d = \mathrm{diag}(x_1,x_2,\ldots,x_n) > 0, \qquad \hat{Y}_d = \mathrm{diag}(y_1,y_2,\ldots,y_n) > 0. \tag{C.2}$$

The integration in (C.1) goes over $\hat{T}\in \mathsf{Gl}(\mathsf{n},\mathcal{C})/\mathsf{U}(1)\times\cdots\times\mathsf{U}(1)$, i.e. complex matrices with real positive diagonal elements, and $\mathrm{d}\mu(\hat{T},\hat{T}^\dagger)$ is the corresponding measure. Note, however that the integrand is not changed if we consider $\hat{T}\in\mathsf{Gl}(\mathsf{n},\mathcal{C})$ rather than restricting it to the coset space as above. Therefore, if we use the invariant Haar's measure $\mathrm{d}\mu_{\mathrm{H}}(\hat{T})$ for the full group, the result of integration will be the same up to a constant factor.

Let us look at the integral in (C.1) as a function $\Phi(X_d,Y_d)$. We start with considering a pair of $n\times n$ Hermitian positive definite matrices $\hat{X}^{(1)},\hat{X}^{(2)}$, and another such pair $\hat{A},\hat{B}$. Introduce the Laplace operator $D_{X^{(1)},X^{(2)}}$ acting on such matrices as

$$D_{X^{(1)},X^{(2)}} = \frac{1}{2}\sum_{i\leqslant i,j\leqslant n}\frac{\partial^2}{\partial(\mathrm{Re}X_{ij}^{(1)})\,\partial(\mathrm{Im}X_{ji}^{(2)})}. \tag{C.3}$$

Then we construct a function $W(X^{(1)},X^{(2)},A,B)$ with the property

$$D_{X^{(1)},X^{(2)}}W(X^{(1)},X^{(2)},A,B) = \mathrm{Tr}(AB)W(X^{(1)},X^{(2)},A,B). \tag{C.4}$$

In particular, the following function

$$W(X^{(1)},X^{(2)},A,B) = \exp[-\tfrac{1}{2}\mathrm{Tr}(X^{(1)}A+BX^{(2)})] \tag{C.5}$$

satisfies the equation (C.4) as can be checked by direct calculations. Let us now use a possibility [18] to parameterize $A = T_YY_dT_Y^\dagger$ and $B = (T_Y^\dagger)^{-1}Y_dT_Y^{-1}$ in equation (C.4), as well as $\hat{X}^{(1)} = T_XX_dT_X^\dagger$, $\hat{X}^{(2)} = (T_X^\dagger)^{-1}X_d(T_X)^{-1}$. Here both T_X, T_Y belong to the coset space $\mathsf{Gl}(\mathsf{n},\mathcal{C})/\mathsf{U}(1)\times\cdots\times\mathsf{U}(1)$.

We obtain

$$D_{X^{(1)},X^{(2)}} \exp -\tfrac{1}{2}\,\mathrm{Tr}([X^{(1)}T_Y Y_d T_Y^\dagger + (T_Y^\dagger)^{-1} Y_d T_Y^{-1} X^{(2)}])$$
$$= \mathrm{Tr}(Y_d^2) \exp[-\tfrac{1}{2}\,\mathrm{Tr}(X_d[\tilde{T} Y_d \tilde{T}^\dagger + (\tilde{T}^\dagger)^{-1} Y_d \tilde{T}^{-1}])] \tag{C.6}$$

where we introduced the matrix $\tilde{T} = T_X^\dagger T_Y$ belonging to the group $\mathsf{Gl}(\mathsf{n}, \mathcal{C})$.

Now we use that (i) the integration over complex matrices T_Y commutes with the Laplace operator $D_{X^{(1)},X^{(2)}}$, and (ii) for any fixed T_X the result of integrating the right-hand side of (C.6) over T_Y does not depend on T_X due to the possibility of using the invariant measure $\mathrm{d}\mu_{\mathrm{H}}(T_Y) = \mathrm{d}\mu_{\mathrm{H}}(\tilde{T})$. We therefore conclude that the matrix function $\Phi(X_d, Y_d)$ defined by the integral equation (C.1) satisfies the following differential equation:

$$D_{X^{(1)},X^{(2)}} \Phi(X_d, Y_d) = \mathrm{Tr}(Y_d^2)\Phi(X_d, Y_d). \tag{C.7}$$

To derive the explicit formula (equation (C.1)) for the matrix function $\Phi(X_d, Y_d)$ we apply the method proposed by Guhr and Wettig [20] duly modified. We notice that when passing from $X^{(1)}, X^{(2)}$ to the 'angular' coordinates T_X and 'radial' coordinates X_d the radial part of the Jacobian is given by $J(X_d) = \Delta^2(x_1^2, \ldots, x_n^2)\prod_{i=1}^n x_i$. This is enough to ensure that the radial part D_{X_d} of the Laplace operator $D_{X^{(1)},X^{(2)}}$ must have the following expression:

$$D_{X_d} = \frac{1}{J(x)}\sum_{i=1}^n \partial_i J(x) \partial_i, \qquad J(x) = \Delta^2(x_1^2, \ldots, x_n^2)\prod_{i=1}^n x_i. \tag{C.8}$$

Guhr and Wettig noted [20] that the radial part D_{X_d} of such form is, in fact, separable. It means that for an arbitrary function $f(x_1, x_2, \ldots, x_n)$ the following identity holds:

$$D_{X_d}\frac{f(x_1, \ldots, x_n)}{\Delta(x_1^2, \ldots, x_n^2)} = \frac{1}{\Delta(x_1^2, \ldots, x_n^2)}\sum_{k=1}^n \left(\frac{\partial^2}{\partial x_k^2} + \frac{1}{x_k}\frac{\partial}{\partial x_k}\right)\frac{f(x_1, \ldots, x_n)}{\Delta(x_1^2, \ldots, x_n^2)}. \tag{C.9}$$

The separability of the operator D_{X_d} enables us to solve the differential equation (C.7) and to prove formula (C.1). The remaining parts of the proof given in the appendix B of [18] hold their validity without any modifications.

References

[1] Efetov K B 1983 *Adv. Phys.* **32** 53
Efetov K B 1997 *Supersymmetry in Disorder and Chaos* (Cambridge: Cambridge University Press)
[2] Mirlin A D 2000 *Phys. Rep.* **36** 259
[3] Wegner F 1979 *Z. Phys.* **36** 207
[4] Schäfer L and Wegner F 1980 *Z. Phys.* B **38** 113
[5] Pruisken A M M and Schäfer L 1982 *Nucl. Phys.* B **200** 20
[6] Spencer T and Zirnbauer M R 2004 *Commun. Math. Phys.* **252** 167 (*Preprint* math-ph/0410032)
[7] Verbaarschot J J M, Weidenmüller H A and Zirnbauer M R 1985 *Phys. Rep.* **129** 367
[8] Fyodorov Y V and Sommers H-J 1997 *J. Math. Phys.* **38** 1918
[9] Verbaarschot J J M and Wettig T 2000 *Annu. Rev. Nucl. Part. Sci.* **50** 343
[10] Fyodorov Y V 1995 *Mesocopic Quantum Physics (Les Houches Session* LXI, 1994) ed E Akkemans *et al* (Amsterdam: Elsevier) p 493
[11] Zirnbauer M R 2004 The supersymmetry method of random matrix theory *Preprint* math-ph/0404057
[12] Fyodorov Y V 2002 *Nucl. Phys.* B **621** (PM) 643
[13] Fyodorov Y V and Mirlin A D 1991 *Phys. Rev. Lett.* **67** 2405
Fyodorov Y V and Mirlin A D 1994 *Int. J. Mod. Phys.* B **8** 3795
[14] Zirnbauer M R 1996 *J. Math. Phys.* **37** 4986 (see the end of section III.E, on page 5006)
[15] Fyodorov Y V and Strahov E 2002 *Nucl. Phys.* B **630** (PM) 453
[16] Gradshteyn I S and Ryzyk I M 2000 *Table of Integrals, Series and Products* 6th edn, ed A Jeffrey (San Diego, CA: Academic) ((a) formula 8.421 and definitions 8.405; (b) formula 6.631.4)
[17] Zirnbauer M R 2004 Symmetry classes in random matrix theory *Preprint* math-ph/0404058

[18] Fyodorov Y V and Strahov E 2002 *Nucl. Phys.* B **647** (PM) 581
[19] Itzykson C and Zuber J B 1980 *J. Math. Phys.* **21** 411
Harish-Chandra 1956 *Proc. Natl Acad. Sci.* **42** 252
[20] Guhr T and Wettig T 1996 *J. Math. Phys.* **37** 6395
Berezin F A and Karpelevich F I 1958 *Dokl. Akad. Nauk SSSR* **118** 9
[21] Jackson A D, Sener M K and Verbaarschot J J M 1996 *Phys. Lett.* B **387** 355
[22] Guhr T and Kohler H 2002 *J. Math. Phys.* **43** 2707
[23] Nishioka N, Verbaarschot J J M, Weidenmüller H A and Yoshida S 1986 *Ann. Phys.* **172** 67–99
[24] Disertori M, Pinson H and Spencer T 2002 *Commun. Math. Phys.* **232** 83
[25] Fyodorov Y V, unpublished